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Second level semi-degenerate fields in \mathcal{W}_3 Toda theory: matrix element and differential equation

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ABSTRACT: In a recent study we considered \mathcal{W}_3 Toda 4-point functions that involve matrix elements of a primary field with the highest-weight in the *adjoint* representation of \mathfrak{sl}_3 . We generalize this result by considering a semi-degenerate primary field, which has one null vector at level two. We obtain a sixth-order Fuchsian differential equation for the conformal blocks. We discuss the presence of multiplicities, the matrix elements and the fusion rules.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory

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1 Introduction

The \mathcal{W}_N theories [1, 2] are 2D conformal field theories (CFT) with current algebra \mathcal{W}_N , the representations of which are related to the weights of the Lie algebra \mathfrak{sl}_N (for a review, we refer the reader to [3]). The Virasoro algebra coincides with the \mathcal{W}_2 algebra and is a sub-algebra of $\mathcal{W}_{N>2}$. In the case of Virasoro CFT [4], all correlation functions can be expressed in terms of correlation functions of primary fields. This is not true for the $\mathcal{W}_{N>2}$ theory because the symmetry constraints are no longer sufficient to fix the operator product expansion or, equivalently, the matrix elements involving general states. In order to compute \mathcal{W}_N correlation functions one needs to find additional conditions besides the ones coming from the current algebra. Typically, these additional conditions can originate from the presence of a \mathcal{W}_N primary field, the null-state, at a certain level of a given representation module. This is the reason why only \mathcal{W}_N correlation functions involving a set of particular fields can be computed. For a general value of the central charge a \mathcal{W}_N representation module can contain up to $N - 1$ null-states: a field is said to be fully- or semi- degenerate if the corresponding module contains $N - 1$ or less null states.

So far, the studies about \mathcal{W}_N theories focused mostly on \mathcal{W}_N correlation functions that contain level-1 semi-degenerate fields [5, 6]. These fields are associated to the (*anti*-)fundamental representation of \mathfrak{sl}_N and we will refer to them as (anti)-fundamental fields. In [7] it was shown that all matrix elements that contain at least one (anti)-fundamental field can be computed explicitly. The interest about this special sector of \mathcal{W}_N was also greatly motivated by the fact that, via the AGT correspondence [8–10], the corresponding conformal blocks are related to the instanton calculus in Ω -deformed $\mathcal{N} = 2$ SUSY $SU(N)$ quiver gauge theories. On the symmetry level this relation is a consequence of the fact the \mathcal{W}_N (and many others) chiral symmetry algebras can be obtained from a special toroidal algebras having simple action on the cohomologies of instanton moduli spaces [11]. This connection, in turn, reveals a deep integrable structure of \mathcal{W}_N theory [12, 13] (for explicit example see, e.g. [14]). In particular, the computation of the matrix elements of the semi-degenerate fundamental fields in the integrable basis [12] gives a nice combinatorial representation for the conformal blocks. Hence, an interesting problem is whether or not one can extend this connection to a larger set of \mathcal{W}_N correlation functions. The first question in this direction is whether there exists any generalization of the semi-degenerate fundamental field, such that the corresponding matrix elements can be also constructed explicitly via \mathcal{W}_N symmetry constraints.

In a recent study [15] we considered a \mathcal{W}_3 Toda 4-point functions that involve a fully-degenerate primary field in the fundamental representation of \mathfrak{sl}_3 and a fully-degenerate primary field in the *adjoint* representation of \mathfrak{sl}_3 . This latter field has *two* null-states at the second level of the associated representation module. In [15], we showed that the associated conformal block satisfy a fourth-order Fuchsian differential equation and we discussed the role of multiplicities that appear in this theory. In this paper we generalize these result by considering the 4-point conformal block that involve, besides the fully-degenerate primary field in the fundamental representation, a semi-degenerate primary field with *one* null state at level two. The associated local correlation functions have been considered in [6] where their expressions in term of four-dimensional integrals were given. Here we show that this conformal block obeys a sixth-order Fuchsian differential equation. We compute the matrix elements of the semi-degenerate field, between two arbitrary descendant states and we verify that the series expansion of the conformal block constructed from these matrix elements agrees with the differential equation.

The paper is organized as follows. In section 2, we recall basic facts regarding \mathcal{W}_3 conformal field theory and introduce the \mathcal{W}_3 conformal block function. In 3 we discuss the null-vector conditions for the degenerate and semi-degenerate primary fields. In 4, we focus on a specific 4-point correlation function with one semi-degenerate level=2 and one fully degenerate field in the fundamental \mathfrak{sl}_3 representation. Here show that the corresponding conformal blocks obeys a sixth-order Fuchsian differential equation. We proceed by discussing the matrix elements of the semi-degenerate level-2 field between two arbitrary descendant states and compare the results of the differential equation with the explicit construction of the conformal blocks in terms of the matrix elements. In 6, we present our conclusion and discuss some open problems. In appendix A, we give some technical details of the derivation of the differential equation.

2 \mathcal{W}_3 chiral algebra, representation modules and conformal blocks

We briefly introduce the \mathcal{W}_3 chiral symmetry algebra and its representation theory. We use the same notations and normalization conventions as in [15].

\mathcal{W}_3 algebra. The \mathcal{W}_3 is an associative algebra generated by the modes L_n and W_n of the spin-2 energy-momentum tensor $\mathcal{T}(z)$ and of the spin-3 holomorphic field $\mathcal{W}(z)$. The full \mathcal{W}_3 algebra is given by the following commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad [L_m, W_n] = (2m - n)W_{m+n}, \quad (2.1)$$

and

$$\begin{aligned} [W_m, W_n] = & \frac{1}{3}(m - n)\Lambda_{m+n} \\ & + \left(\frac{22 + 5c}{48} \right) \left(\frac{m - n}{30} \right) (2m^2 - mn + 2n^2 - 8) L_{m+n} \\ & + \left(\frac{22 + 5c}{48} \right) \left(\frac{c}{3 \cdot 5!} \right) (m^2 - 4)(m^3 - m) \delta_{m+n,0}, \end{aligned} \quad (2.2)$$

where Λ_m are the modes of the quasi-primary field $\Lambda =: \mathcal{T}^2 : - \frac{3}{10} \partial^2 \mathcal{T}$ and the colons $: :$ stand for normal-ordering. Explicitly,

$$\Lambda_m = \sum_{p \leq -2} L_p L_{m-p} + \sum_{p \geq -1} L_{m-p} L_p - \frac{3}{10}(m + 2)(m + 3)L_m. \quad (2.3)$$

In (2.2) we assume the following normalisation for the current $\mathcal{W}(z)$ 2-point function:

$$\langle \mathcal{W}(1)\mathcal{W}(0) \rangle = \frac{c}{3}\eta \quad \text{with} \quad \eta \stackrel{\text{def}}{=} \left(\frac{22 + 5c}{48} \right). \quad (2.4)$$

The parametrisation of the \mathcal{W}_3 central charge c , commonly used in the Toda field theory literature, is

$$c = 2 + 24 Q^2, \quad Q = b + \frac{1}{b}. \quad (2.5)$$

\mathcal{W}_3 primary fields. A \mathcal{W}_3 primary field $\Phi_{\vec{\alpha}}(z)$ is completely characterized by the pair of quantum numbers (h, q) , respectively its conformal dimension (h) and W_0 eigenvalue (q) . It is labelled by a vector $\vec{\alpha}$ in the space spanned by the fundamental \mathfrak{sl}_3 weights,

$$\vec{\alpha} = \alpha_1 \vec{\omega}_1 + \alpha_2 \vec{\omega}_2, \quad (2.6)$$

where the standard \mathfrak{sl}_3 conventions are used:

$$\vec{\omega}_1 = \sqrt{\frac{2}{3}}(1, 0); \quad \vec{\omega}_2 = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right); \quad (2.7)$$

$$\vec{e}_1 = 2\vec{\omega}_1 - \vec{\omega}_2; \quad \vec{e}_2 = -\vec{\omega}_1 + 2\vec{\omega}_2; \quad \vec{\rho} = \vec{\omega}_1 + \vec{\omega}_2, \quad (2.8)$$

$$\vec{h}_1 = \vec{\omega}_1; \quad \vec{h}_2 = \vec{\omega}_1 - \vec{e}_1; \quad \vec{h}_3 = \vec{\omega}_1 - \vec{e}_1 - \vec{e}_2. \quad (2.9)$$

In terms of the parameters

$$x_i = (Q\vec{\rho} - \vec{\alpha}) \cdot \vec{h}_i, \quad i = 1, 2, 3, \quad (2.10)$$

one has

$$h = Q^2 + x_1x_2 + x_1x_3 + x_2x_3 \quad \text{and} \quad q = i x_1x_2x_3. \quad (2.11)$$

The fields $\Phi_{\vec{\alpha}}$ are normalized in such a way that the 2-point correlation function $\langle \Phi_{\vec{\alpha}}^*(z) \Phi_{\vec{\alpha}}(0) \rangle$, the $\Phi_{\vec{\alpha}}^*(z) \stackrel{\text{def}}{=} \Phi_{2Q\vec{\rho} - \vec{\alpha}}(z)$ satisfies:

$$\lim_{z \rightarrow \infty} z^{2h} \langle \Phi_{\vec{\alpha}}^*(z) \Phi_{\vec{\alpha}}(0) \rangle = 1. \quad (2.12)$$

In the following we will also use the notation $\vec{\alpha}^* \stackrel{\text{def}}{=} 2Q\vec{\rho} - \vec{\alpha}$.

\mathcal{W}_3 representation module. The representation module $\mathcal{V}_{\vec{\alpha}}$ associated to $\Phi_{\vec{\alpha}}$ is spanned by the basis states,

$$\Phi_{\vec{\alpha}}^{(I)} \stackrel{\text{def}}{=} \mathcal{L}_I \Phi_{\vec{\alpha}} \stackrel{\text{def}}{=} L_{-i_m} \cdots L_{-i_1} W_{-j_n} \cdots W_{-j_1} \Phi_{\vec{\alpha}}, \quad (2.13)$$

where the sets of positive integers

$$I = \{i_m, \dots, i_1; j_n, \dots, j_1\} \quad \text{with} \quad i_m \geq \dots \geq i_1 \geq 1, \quad j_n \geq \dots \geq j_1 \geq 1, \quad (2.14)$$

are normal-ordered, The symbol \emptyset will be used when no modes L_m or W_n are present. For instance, set $I = \{\emptyset; 3, 1\}$, then $\Phi_{\vec{\alpha}}^{(I)} = W_{-3}W_{-1}\Phi_{\vec{\alpha}}$. The descendant fields $\Phi_{\vec{\alpha}}^{(I)}$ have conformal dimension $h + |I|$, where $|I| = i_1 + i_2 + \dots + j_1 + j_2 + \dots$ is called the level. Any \mathcal{W}_3 highest-weight representation is spanned by the states (2.13) [3]. We also refer to the appendix of [15] where these properties are reviewed in the same notations and conventions adopted here.

The Shapovalov matrix of inner products. The Shapovalov matrix H , whose ij -element H_{ij} is the scalar product of the states $|\mathcal{L}_I \Phi_{\vec{\alpha}}\rangle$ and $|\mathcal{L}_J \Phi_{\vec{\alpha}}\rangle$

$$H_{IJ}(\vec{\alpha}) = \langle \mathcal{L}_I \Phi_{\vec{\alpha}} | \mathcal{L}_J \Phi_{\vec{\alpha}} \rangle, \quad (2.15)$$

has a block-diagonal structure, $H(\vec{\alpha}) = \text{diag} \left(H^{(0)}(\vec{\alpha}), H^{(1)}(\vec{\alpha}), H^{(2)}(\vec{\alpha}), \dots \right)$, where the elements of the i -th block, $H^{(i)}(\vec{\alpha})$, are the scalar products of the level- i descendants. These elements can be computed using the commutation relations (2.2), (2.1). By definition, $H^{(0)}(\vec{\alpha}) = 1$. The explicit forms of $H^{(1)}(\vec{\alpha})$ and $H^{(2)}(\vec{\alpha})$ can be found for instance in the appendix of [15].

The matrix elements. The matrix elements of general descendant fields are defined by:

$$\Gamma_{I,J,K} \left(L, M, R \right) = \frac{\langle (\Phi_L^*)^{(I)} | \Phi_M^{(J)}(1) \Phi_R^{(K)}(0) \rangle}{\langle (\Phi_L^*) | \Phi_M(1) \Phi_R(0) \rangle}, \quad (2.16)$$

$$\Gamma'_{I,J,K} \left(L, M, R \right) = \frac{\langle \Phi_L^{(I)} | \Phi_M^{(J)}(1) \Phi_R^{(K)}(0) \rangle}{\langle \Phi_L | \Phi_M(1) \Phi_R(0) \rangle}, \quad (2.17)$$

where L, M, R is a short notation for $\vec{\alpha}_L, \vec{\alpha}_M, \vec{\alpha}_R$ and $\Phi_X^{(I)}$ ($X = L, M, R$) is defined in (2.13).

Degenerate representations and fusion rules. A \mathcal{W}_3 fully-degenerate representation $\mathcal{V}_{r_1 r_2 s_1 s_2}$ is associated with the primary field $\Phi_{\vec{\alpha}_{r_1 r_2 s_1 s_2}}(z)$ with

$$\vec{\alpha}_{r_1 r_2 s_1 s_2} = b \left((1 - r_1) \vec{\omega}_1 + (1 - r_2) \vec{\omega}_2 \right) + \frac{1}{b} \left((1 - s_1) \vec{\omega}_1 + (1 - s_2) \vec{\omega}_2 \right), \quad (2.18)$$

where r_1, r_2, s_1, s_2 are positive integers. In the following we use the notation:

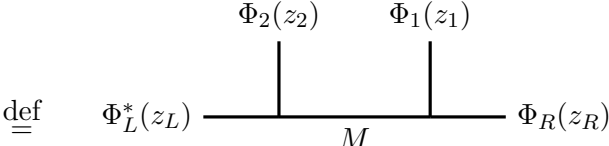
$$\Phi_{r_1 r_2 s_1 s_2} \stackrel{\text{def}}{=} \Phi_{\vec{\alpha}_{r_1 r_2 s_1 s_2}}. \quad (2.19)$$

The representation $\mathcal{V}_{r_1 r_2 s_1 s_2}$ exhibits two independent null-states at levels $r_1 s_1$ and $r_2 s_2$ and the fusion products of $\mathcal{V}_{r_1 r_2 s_1 s_2}$ with a general \mathcal{W}_3 irreducible module $\Phi_{\vec{\alpha}}$ takes the form

$$\mathcal{V}_{r_1 r_2 s_1 s_2} \times \mathcal{V}_{\vec{\alpha}} = \sum_{\vec{h}_r, \vec{h}_s} \mathcal{V}_{\vec{\alpha} - b\vec{h}_r - b^{-1}\vec{h}_s}, \quad (2.20)$$

where h_r and h_s are the weights of the \mathfrak{sl}_3 representation with highest-weight $(r_1 - 1) \vec{\omega}_1 + (r_2 - 2) \vec{\omega}_2$ and $(s_1 - 1) \vec{\omega}_1 + (s_2 - 2) \vec{\omega}_2$ respectively.

\mathcal{W}_3 conformal blocks. The conformal block $\mathcal{B}_M \left(L, 2, 1, R \right) (\{z_i\})$ with internal fusion channel Φ_M can be represented as the following *comb diagram*:

$$\mathcal{B}_M \left(L, 2, 1, R \right) (\{z_i\}) \stackrel{\text{def}}{=} \langle \Phi_L^*(z_L) | \Phi_2(z_2) \Phi_1(z_1) \Phi_R(z_R) \rangle$$


$$\stackrel{\text{def}}{=} \quad \Phi_L^*(z_L) \text{ --- } \overset{\Phi_2(z_2)}{\text{---}} \underset{M}{\text{---}} \overset{\Phi_1(z_1)}{\text{---}} \text{ --- } \Phi_R(z_R) \quad (2.21)$$

Invariance under global conformal transformations implies:

$$\begin{aligned} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_i\}) &= (z_L - z_1)^{-2h_1} (z_L - z_R)^{h_1 - h_R + h_2 - h_L} (z_L - z_2)^{h_1 + h_R - h_2 - h_L} \times \\ &\times (z_2 - z_R)^{-h_1 - h_R - h_2 + h_L} \mathcal{B}_M \left(L, 2, 1, R \right) (z), \end{aligned} \quad (2.22)$$

where

$$z = \frac{(z_1 - z_R)(z_2 - z_L)}{(z_1 - z_L)(z_2 - z_R)}. \quad (2.23)$$

The function $\mathcal{B}_M \left(L, 2, 1, R \right) (z)$ is defined by the following expansion

$$\begin{aligned} z^{h_L + h_R - h_M} \mathcal{B}_M \left(L, 2, 1, R \right) (z) &= \\ &= 1 + \sum_{i=1}^{\infty} z^i \sum_{\substack{K, K' \\ |K|=|K'|=i}} \left[H^{(i)} \left(M \right) \right]_{K, K'}^{-1} \Gamma_{\emptyset, \emptyset, K} \left(L, 2, M \right) \Gamma'_{K', \emptyset, \emptyset} \left(M, 1, R \right), \end{aligned} \quad (2.24)$$

where the matrix $H^{(i)}$ and the matrix elements Γ_{I_L, J_M, K_R} and Γ'_{I_L, J_M, K_R} were defined in (2.15) and (2.16).

\mathcal{W}_3 Ward identities. While the three Ward identities associated with the conserved current \mathcal{T} fix the form (2.22), there are other five Ward identities associated with the conserved current \mathcal{W}

$$\sum_{X=L,2,1,R} W_{-2}^{(X)} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = 0, \quad (2.25)$$

$$\sum_{X=L,2,1,R} \left(z_X W_{-2}^{(X)} + W_{-1}^{(X)} \right) \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = 0, \quad (2.26)$$

$$\sum_{X=L,2,1,R} \left(z_X^2 W_{-2}^{(X)} + 2z_X W_{-1}^{(X)} + q_X \right) \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = 0, \quad (2.27)$$

$$\sum_{X=L,2,1,R} \left(z_X^3 W_{-2}^{(X)} + 3z_X^2 W_{-1}^{(X)} + 3z_X q_X \right) \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = 0, \quad (2.28)$$

$$\sum_{X=L,2,1,R}^N \left(z_X^4 W_{-2}^{(X)} + 4z_X^3 W_{-1}^{(X)} + 6z_X^2 q_X \right) \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = 0, \quad (2.29)$$

where the notation $W_{-i}^{(X)}$, $i = 1, 2$ means that the mode W_{-i} is applied to the field $X (= L, 2, 1, R)$ in the conformal block.

3 Null-states in semi- and fully-degenerate representations

3.1 Null-state equations for the fully-degenerate fundamental field

We list here the null-state equations for the fully-degenerate fundamental field $\Phi_{2111}(z)$ that we will need later. The field $\Phi_{2111}(z) = \Phi_{-b\vec{\omega}_1}(z)$ has quantum numbers

$$h = h_1 \stackrel{\text{def}}{=} \frac{1}{3} \left(-3 - 4b^2 \right), \quad q = q_1 \stackrel{\text{def}}{=} \frac{i}{27b} \left(3 + 4b^2 \right) \left(3 + 5b^2 \right). \quad (3.1)$$

This field obeys level-1, level-2 and level-3 null-state conditions:

$$W_{-1} \Phi_{-b\vec{\omega}_1} = \frac{3q_1}{2h_1} L_{-1} \Phi_{-b\vec{\omega}_1}. \quad (3.2)$$

$$W_{-2} \Phi_{-b\vec{\omega}_1} = \left(\frac{12q_1}{h_1(5h_1+1)} L_{-1}^2 - \frac{6q_1(h_1+1)}{h_1(5h_1+1)} L_{-2} \right) \Phi_{-b\vec{\omega}_1}, \quad (3.3)$$

and

$$W_{-3} \Phi_{-b\vec{\omega}_1} = \left(\frac{16q_1}{h_1(h_1+1)(5h_1+1)} L_{-1}^3 - \frac{12q_1}{h_1(5h_1+1)} L_{-1} L_{-2} + \frac{3q_1(h_1-3)}{2h_1(5h_1+1)} L_{-3} \right) \Phi_{-b\vec{\omega}_1}. \quad (3.4)$$

3.2 Semi-degenerate representation at level-two

In this paper we are interested in the case of general central charge, in particular when $b^2 \notin \mathbb{Q}$. Consider a field $\Phi_{r_1 r_2 s_1 s_2}$ with r_1 and s_1 positive integers and $r_2 \notin \mathbb{N}^+$. In the representation module there is one null-vector at level $r_1 s_1$ with quantum numbers (h, q)

coinciding with the ones of the primary operator $\Phi_{r'_1 r'_2 s_1 s_2}$ with $r'_1 = -r_1$ and $r'_2 = r_2 + r_1$, see [16] and references therein. Here we want to focus on the semi-degenerate field at level two $\Phi_{-b\bar{\omega}_1 + s\bar{\omega}_2}$, which corresponds to $\Phi_{r_1 r_2 s_1 s_2}$ with $r_1 = 2$ and $r_2 = 1 - s$. In [17], the explicit expression of the null vector at level r is associated to the field $\Phi_{(1-r)b\bar{\omega}_1 + s\bar{\omega}_2}$ has been given. For sake of clarity, we derive below, by using our conventions, the same result for the level two null-vector associated to the field $\Phi_{-b\omega_1 + s\omega_2}$.

Let consider a general module $\mathcal{V}_{\vec{\alpha}}$ with quantum number (h, q) . At the second level one has five fields $\Phi^{(I)}$ with $|I| = 2$:

$$|I| = 2 : \quad I = \{2; \emptyset\}, \{\emptyset; 2\}, \{1, 1; \emptyset\}, \{1; 1\}, \{\emptyset; 1, 1\}. \quad (3.5)$$

From (2.13), a general field Ψ at level two can be written as a linear combination of the above five fields:

$$\Psi_{\vec{\alpha}} \stackrel{\text{def}}{=} \left(c_{\{2; \emptyset\}} L_{-2} + c_{\{\emptyset; 2\}} W_{-2} + c_{\{1, 1; \emptyset\}} L_{-1}^2 + c_{\{1; 1\}} L_{-1} W_{-1} + W_{-1}^2 \right) \Phi_{\vec{\alpha}}, \quad (3.6)$$

where the global normalisation has been fixed by setting the coefficient of $W_{-1}^2 \Phi_{\vec{\alpha}}$ to one.

We assume Ψ to be a \mathcal{W}_3 primary field, i.e.

$$W_0 \Psi_{\vec{\alpha}} = q' \Psi_{\vec{\alpha}}, \quad (3.7)$$

$$L_1 \Psi_{\vec{\alpha}} = L_2 \Psi_{\vec{\alpha}} = 0. \quad (3.8)$$

We will see below that the above relations, in particular the fact that W_0 is diagonalizable, give the correct result. A discussion about the fact that W_0 need not be diagonalizable can be found for instance in [18]. The annihilation by W_1 and W_2 and higher generators follows from the \mathcal{W}_3 commutation relations.

The requirement (3.7) is satisfied if

$$Mu = q'u, \quad (3.9)$$

where c is the vector $u = (u_{\{2; \emptyset\}}, u_{\{\emptyset; 2\}}, u_{\{1, 1; \emptyset\}}, u_{\{1; 1\}}, u_{\{\emptyset; 1, 1\}})$ and the matrix M reads

$$M = \begin{pmatrix} q & \frac{4h}{3} & 0 & 0 & 2q \\ 4 & q & 2 & 0 & \frac{2-c+32h}{48} \\ 0 & \frac{2}{3} & q & \frac{2-c+32h}{48} & 0 \\ 0 & 0 & 4 & q & \frac{18-c+32h}{24} \\ 0 & 0 & 0 & 2 & q \end{pmatrix}. \quad (3.10)$$

The requirement (3.8) is equivalent to the condition

$$Nu = 0, \quad (3.11)$$

where

$$N = \begin{pmatrix} 4h + \frac{c}{2} & 6q & 6h & 9q & \frac{5h(2-c+32h)}{48} \\ 3 & 0 & 2(2h+1) & 3q & \frac{2-c+32h}{16} \\ 0 & 4 & 0 & 2(h+1) & 6q \end{pmatrix}. \quad (3.12)$$

We consider now the field $\Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}$ that has quantum numbers

$$h = h_2 \stackrel{\text{def}}{=} \frac{3(s-b) + b(s-2b)^2}{3b}, \quad q = q_2 \stackrel{\text{def}}{=} i \frac{(b+s)(4b^2-2bs+3)(5b^2-bs+3)}{27b^2}. \quad (3.13)$$

We have found that the vector

$$u^{\text{sing}} = \left(-\frac{(1+2b^2-bs)(2+2b^2-bs)}{3}, -\frac{i(1+3b^2)(3+4b^2-2bs)}{6b}, -\frac{(3+b^2-2bs)(3+7b^2-2bs)}{36b^2}, -\frac{i(3+4b^2-2bs)}{3b}, 1 \right) \quad (3.14)$$

satisfies the conditions (3.9) and (3.11) with the following eigenvalue

$$q'_{\text{sing}} = \frac{i(5b-s)(3+4b^2-2bs)(-3+b^2+bs)}{27b^2}. \quad (3.15)$$

One can directly verify that q'_{sing} is indeed the W_0 eigenvalue of the field $\Phi_{3b\omega_1+(-2+s)b\omega_2}$, see the discussion at the beginning of the section. We have therefore shown that the field $\Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}$ has a singular-state at level two. Decoupling of the singular vector $\Psi_{-b\vec{\omega}_1+s\vec{\omega}_2} = 0$ yields the following null-state condition

$$\sum_{|I|=2} u_I^{\text{sing}} \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}^{(I)} = 0. \quad (3.16)$$

In a similar way, a similar null-vector condition for the other semi-degenerate field at level two, $\Phi_{s\vec{\omega}_1-b\vec{\omega}_2}$ can be obtained. The remaining two other cases are obtained by replacing $b \rightarrow 1/b$.

4 Sixth-order differential equation

We consider here the conformal block defined in (2.21) with the following identifications

$$\Phi_2 = \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}, \quad \Phi_1 = \Phi_{-b\vec{\omega}_1}. \quad (4.1)$$

We use a shorter notation for the conformal block, $\mathcal{B}_M \left(L, 2, 1, R \right) (z) \rightarrow \mathcal{B}_M(z)$:

$$\mathcal{B}_M(z) \stackrel{\text{def}}{=} \Phi_L^*(\infty) \begin{array}{c} \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}(1) \quad \Phi_{2111}(z) \\ | \qquad \qquad | \\ \hline M \\ \hline \end{array} \Phi_R(0) \quad (4.2)$$

In the following we present in full detail the computation of this conformal block. The case in which $\Phi_2 = \Phi_{s\vec{\omega}_1-b\vec{\omega}_2}$ will be briefly discussed in section 5.2.3.

Let us sketch the procedure to obtain the differential equation satisfied by the conformal block defined above. At the beginning we have

$$9 \text{ unknown functions : } \mathcal{B}_M(z) \text{ and } W_{-i}^{(X)} \mathcal{B}_M(z) \quad i = 1, 2, X = L, 2, 1, R.$$

We recall that the function $W_{-i}^{(X)} \mathcal{B}_M(z)$ is the conformal block that involves the descendant $W_{-i} \Phi_X$ (for some explicit examples see the appendix A).

Using the 8 equations coming from the 5 Ward identities (2.25)–(2.29) plus the 3 null-state conditions (3.2), (3.3) and (3.4), we arrive to an equation of the form

$$[\partial_z^3 + \dots] \mathcal{B}_M(z) = \dots W_{-1}^{(2)} \mathcal{B}_M(z), \quad (4.3)$$

where on the l.h.s. the dots stand for a certain linear combination of differential operators, for instance $z^{-1} \partial_z^2$, $(z-1)^{-2} \partial_z^2$, $(z-1)^2 \partial_z$, $(z-1)^{-2}$ etc., acting on $\mathcal{B}_M(z)$. On the r.h.s. the dots also represent some known linear combination of the factors z^{-3} , $(z-1)^{-3}$, $z^{-2}(z-1)^{-1}$ and $z^{-1}(z-1)^{-2}$ multiplying $W_{-1}^{(2)} \mathcal{B}_M(z)$. We notice that if we had identified $\Phi_2 = \Phi_{s\bar{\omega}_1}$, which is semi-degenerate at level 1, we could have used the relation $W_{-1} \Phi_{s\bar{\omega}_1}(z) \propto \partial_z \Phi_{s\bar{\omega}_1}(z)$ in the eq. (4.3) and obtained the third-order generalized hypergeometric differential equation of [5]. For general \mathcal{W}_N conformal blocks, containing one fully-degenerate and one semi-degenerate fundamental field, one obtains a N -order generalized hypergeometric equation [5]. A detailed study of the \mathcal{W}_4 theory can be found in [19]. In the case under consideration here, eq. (4.2), the null-vector condition (3.16) relates states at level 2. We repeat the procedure leading to (4.3), this time for the fields:

$$W_{-1}^{(2)} \mathcal{B}_M(z) \quad \text{plus} \quad 8 \quad \text{unknown functions:} \quad W_{-i}^{(X)} W_{-1}^{(2)} \mathcal{B}_M(z) \quad i = 1, 2, X = L, 2, 1, R.$$

Notice that in total we have 17 unknown functions. Again we use the 8 equations expressing the three null-state conditions (3.2), (3.3) and (3.4) and the five Ward identities applied to the function $W_{-1}^{(2)} \mathcal{B}_M(z)$. With respect to the eqs. (2.25)–(2.29), these five Ward identities have additional terms that originate from the fact that $W_{-1}^{(2)} \mathcal{B}_M(z)$ involves a descendant state. Restoring the dependence on the coordinates z_L, z_2, z_1, z_R , the modified Ward identities take the form

$$\sum_{X=L,2,1,R} W_{-2}^{(X)} W_{-1}^{(2)} \mathcal{B}_M(\{z_X\}) = 0, \quad (4.4)$$

$$\sum_{X=L,2,1,R} \left(z_X W_{-2}^{(X)} + W_{-1}^{(X)} \right) W_{-1}^{(2)} \mathcal{B}_M(\{z_X\}) = 0, \quad (4.5)$$

$$\begin{aligned} \sum_{X=L,2,1,R} \left(z_X^2 W_{-2}^{(X)} + 2z_X W_{-1}^{(X)} + q_X \right) W_{-1}^{(2)} \mathcal{B}_M(\{z_X\}) + \\ + \kappa \partial_{z_2} \mathcal{B}_M(\{z_X\}) = 0, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \sum_{X=L,2,1,R} \left(z_X^3 W_{-2}^{(X)} + 3z_X^2 W_{-1}^{(X)} + 3z_X q_X \right) W_{-1}^{(2)} \mathcal{B}_M(\{z_X\}) + \\ + 3 z_2 \kappa \partial_{z_2} \mathcal{B}_M(\{z_X\}) + h_2 \kappa \mathcal{B}_M(\{z_X\}) = 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \sum_{X=L,2,1,R}^N \left(z_X^4 W_{-2}^{(X)} + 4z_X^3 W_{-1}^{(X)} + 6z_X^2 q_X \right) W_{-1}^{(2)} \mathcal{B}_M(\{z_X\}) + \\ + 6 z_2^2 \kappa \partial_{z_2} \mathcal{B}_M(\{z_X\}) + 4 h_2 \kappa \mathcal{B}_M(\{z_X\}) = 0, \end{aligned} \quad (4.8)$$

where

$$\kappa = \frac{1}{48} (2 - c + h_2), \quad (4.9)$$

and the dimension h_2 is given in (3.13). This time we obtain the relation of the type

$$[\partial_z^3 + \dots] W_{-1}^{(2)} \mathcal{B}_M(z) = \dots [W_{-1}^2]^{(2)} \mathcal{B}_M(z). \quad (4.10)$$

Note that the above manipulations are valid for a general field $\Phi_2(x)$. We can now use the fact that $\Phi_2 = \Phi_{-b\vec{\omega}_1 + s\vec{\omega}_2}$ and use (3.16) in order to express $[W_{-1}^2]^{(2)} \mathcal{B}_M(z)$ in terms of the functions $\mathcal{B}_M(z)$ and $W_{-1}^{(2)} \mathcal{B}_M(z)$

$$[W_{-1}^2]^{(2)} \mathcal{B}_M(z) = \dots \mathcal{B}_M''(z) + \dots z^{-1} \mathcal{B}_M'(z) + \dots + \dots \partial_z W_{-1}^{(2)} \mathcal{B}_M(z) + \dots z^{-1} W_{-1}^{(2)} \mathcal{B}_M(z) + \dots \quad (4.11)$$

Using the system of equations (4.3), (4.10) and (4.11), we finally obtained a sixth-order differential equation for the function $\mathcal{B}_M(z)$. More details of the calculations are collected in the appendix A. However, the final and explicit expression of the sixth-order differential is too long to be reported here. We discuss instead the results that follow from it.

5 Local exponents and matrix elements

Defining

$$\vec{\alpha}_R = a_{R_1} \vec{\omega}_1 + a_{R_2} \vec{\omega}_2, \quad \vec{\alpha}_L = a_{L_1} \vec{\omega}_1 + a_{L_2} \vec{\omega}_2, \quad (5.1)$$

the conformal block $\mathcal{B}_M(z)$ is a function of five parameters, a_{R_1} , a_{R_2} , a_{L_1} , a_{L_2} and b .

5.1 Local exponents from the differential equation

The Fuchsian differential equation of order six has $2 + 1$ singularities at $0, 1$ and ∞ . We refer the reader to [20] for an exhaustive overview of Fuchsian systems. In Riemann-symbol notation the local exponents ρ_i^0, ρ_i^1 and ρ^∞ , $i = 1, \dots, 6$, associated to the $2 + 1$ singular points $0, 1$ and ∞ can be represented as

$$\left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_1 + 1 & \beta_1 + 1 & \gamma_1 + 1 \\ \alpha_2 & \beta_1 + 2 & \gamma_2 \\ \alpha_2 + 1 & \beta_2 & \gamma_2 + 1 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_3 + 1 & \beta_3 + 1 & \gamma_3 + 1 \end{array} \right\} \quad (5.2)$$

with

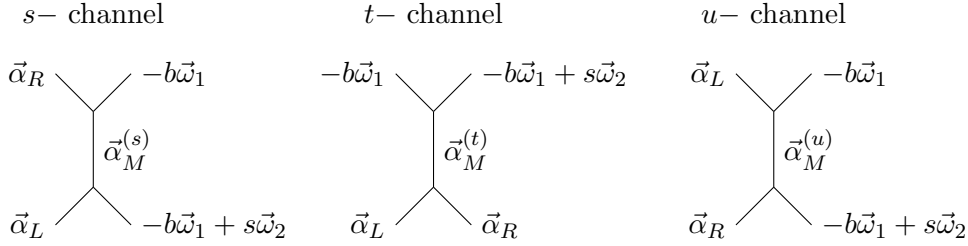
$$\begin{aligned} \alpha_1 &= \frac{1}{3} (2a_{R_1} b + a_{R_2} b), & \alpha_2 &= \frac{1}{3} (3 - a_{R_1} b + a_{R_2} b + 3b^2), & \alpha_3 &= \frac{1}{3} (6 - a_{R_1} b - 2a_{R_2} b + 6b^2), \\ \beta_1 &= \frac{1}{3} (-2b^2 + bs), & \beta_2 &= \frac{1}{3} (3 + 4b^2 + bs), & \beta_3 &= \frac{1}{3} (6 + 7b^2 - 2bs), \\ \gamma_1 &= \frac{1}{3} (a_{L_1} b - a_{L_2} b - 5b^2 - 3), & \gamma_2 &= \frac{1}{3} (-6 + a_{L_1} b + 2a_{L_2} b - 8b^2), & \gamma_3 &= \frac{1}{3} (-2a_{L_1} b - a_{L_2} b - 2b^2). \end{aligned} \quad (5.3)$$

It is easily checked that the above local exponents satisfy the Fuchs identity:

$$\sum_{i=1}^n (\rho_i^0 + \rho_i^1 + \rho_i^\infty) = (k-1) \frac{n(n-1)}{2} = 15, \quad (5.4)$$

specified in our case where Fuchsian equation has order $n = 6$ and number of singularity $2(=k)+1$. It is important to observe that in our Fuchsian equation the fact that there are local exponent different by integers do not imply logarithmic solutions. Indeed, we argue below that the structure of local exponents does not origin from logarithmic features of the theory but from the undetermined matrix elements in the \mathcal{W}_3 algebra.

We can verify that the values of the local exponents can be found by using the fusion products (2.20). To this end we note that the solutions of the differential equation with diagonal monodromies around $0, 1, \infty$ define correspondingly s, t, u conformal blocks associated with the following diagrams:



and that we denote by $\mathcal{B}_M^{(s)}(z)$, $\mathcal{B}_M^{(t)}(z)$ and $\mathcal{B}_M^{(u)}(z)$. The crossing symmetry relation relates

$$\mathcal{B}_M^{(s)}(z) \leftrightarrow \mathcal{B}_M^{(t)}(1-z) \leftrightarrow z^{-2h_1} \mathcal{B}_M^{(u)}(1/z). \quad (5.5)$$

Note, that in order to reconstruct s, t, u fusion rules from the differential equation, one has to take into account factors coming from the fields transformation (non trivial only in $z \rightarrow 1/z$ transformation). Substituting one of these three functions in the differential equation and keeping leading terms in the expansions around $0, 1, \infty$ we find that the exponents α_i , β_i and γ_i , associated respectively to the s -, t -, and u - channels, correspond to the following fusion channels:

- s -channel:

$$\text{Channel 1,2 } (\alpha_1): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R - b\vec{\omega}_1 \quad (5.6)$$

$$\text{Channel 3,4 } (\alpha_2): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R + b \left(\vec{\omega}_1 - \vec{\omega}_2 \right) \quad (5.7)$$

$$\text{Channel 5,6 } (\alpha_3): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R + b\vec{\omega}_2 \quad (5.8)$$

- t -channel:

$$\text{Channel 1,2,3 } (\beta_1): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) - b\vec{\omega}_1 \quad (5.9)$$

$$\text{Channel 4 } (\beta_2): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) + b \left(\vec{\omega}_1 - \vec{\omega}_2 \right) \quad (5.10)$$

$$\text{Channel 5,6 } (\beta_3): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) + b\vec{\omega}_2 \quad (5.11)$$

- u -channel:

$$\text{Channel 1,2 } (\gamma_1): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* - b\vec{\omega}_1 \quad (5.12)$$

$$\text{Channel 3,4 } (\gamma_2): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* + b \left(\vec{\omega}_1 - \vec{\omega}_2 \right) \quad (5.13)$$

$$\text{Channel 5,6 } (\gamma_3): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* + b\vec{\omega}_2 \quad (5.14)$$

We note that as usual (in \mathcal{W}_3 case) in the u -channel we use the conjugated value $\vec{\alpha}_L^*$ which is defined below (2.12).

5.2 Multiplicities and matrix elements

The multiplicities of the local exponents can be argued from the 4-point conformal block expansion (2.24). As explained in [5, 7, 15, 17, 21–23] any matrix element $\Gamma_{I,J,K}$ of three arbitrary \mathcal{W}_3 descendant states, $\Phi_L^{(I)}$, $\Phi_M^{(J)}$, and $\Phi_R^{(K)}$ can be written as linear combinations of the matrix elements

$$\Gamma_{\{\emptyset;\emptyset\},\{\emptyset;\underbrace{1,1,1,\dots}_{p \text{ times}}\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, \vec{\alpha}_M, \vec{\alpha}_R \right), \quad p = 1, 2, 3, \dots \quad (5.15)$$

where the Γ have been defined in (2.16).

5.2.1 Multiplities in the s – and u –channel

Consider for instance the conformal block expansion (2.24) in the s –channel. To determine the first order coefficient one needs to evaluate the two matrix elements:

$$\Gamma_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_M^{(s)}, -b\vec{\omega}_1, \vec{\alpha}_R \right), \quad \Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_M^{(s)} \right). \quad (5.16)$$

If the Γ matrix element, involving the fully-degenerate field $\Phi_{-b\vec{\omega}_1}$ can be evaluated using the null-state equation (3.2):

$$\Gamma_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_M^{(s)}, -b\vec{\omega}_1, \vec{\alpha}_R \right) = \frac{3q_1}{2h_1} \left(q_M^{(s)} - q_R - q_1 \right), \quad (5.17)$$

the matrix element $\Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_M^{(s)} \right)$ is determined only if $s = -b$ (and $h_L \neq h_M^{(s)}$), i.e. when one of the fields entering the matrix elements is a fully-degenerate field in the adjoint representation. This case was fully considered in [15]. For $s \neq -b$ instead, the condition at second level (3.16) alone is not sufficient to fix this matrix element. This ambiguity is at the origin of the multiplicity of order two of the local exponents α_i . Indeed, let us fix

$$\Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_M^{(s)} \right) = \lambda, \quad (5.18)$$

where λ is an arbitrary constant. This is equivalent, from the point of view of the differential equation, to choose a particular combination in the two-dimensional space of solutions having the same local exponent α_i . On the other hand, the semi-degenerate condition (3.16) allows to express all the other matrix elements (5.15) with $p = 2, 3 \dots$ as functions of (5.18). For instance,

$$\begin{aligned} \Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1,1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_M^{(s)} \right) &= -c_{\{2;\emptyset\}}^{\text{sing}} \left(-h_L + 2h_M^{(s)} + h_2 \right) + \\ &+ c_{\{\emptyset;2\}}^{\text{sing}} \left(q_M^{(s)} + q_L + q_2 + 2\lambda \right) - c_{\{1,1;\emptyset\}}^{\text{sing}} \left((h_L - h_M^{(s)} - h_2 - 1)(h_L - h_M^{(s)} - h_2) \right) - \\ &- c_{\{1;1\}}^{\text{sing}} \left((h_L - h_M^{(s)} - h_2 - 1)\lambda \right). \end{aligned} \quad (5.19)$$

We have checked up to the second order in the expansion (2.24), that the direct computation of the matrix element is in agreement with the results obtained by using the sixth-order differential equation. The same arguments seen for explaining the multiplicities in the s -channel holds for the u -channel.

5.2.2 Multiplicities in the t -channel

One can notice that in the t -channel the structure of the multiplicities is different. Again this can be understood by considering the matrix elements entering in the t -channel expansions. For the fusion channels 1, 2, 3, we have two undetermined matrix elements at first and second order:

$$\Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -2b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_R \right), \quad \Gamma'_{\{\emptyset;\emptyset\},\{\emptyset;1,1\},\{\emptyset;\emptyset\}} \left(\vec{\alpha}_L, -2b\vec{\omega}_1 + s\vec{\omega}_2, \vec{\alpha}_R \right). \quad (5.20)$$

At the level of the differential equation, this corresponds to the fact that, in order to select a function in the three-dimensional space of solutions with local exponent β_1 , we have to fix two parameters. Fixing these two parameters is equivalent to fixing the values of the above matrix elements. Once these two parameters have been set, all the other matrix elements of higher-order can be computed in terms of these two parameters, as we have seen before. This is consistent with the fact that the field $\Phi_{-2b\vec{\omega}_1+s\vec{\omega}_2}$ obeys a null-state condition at order three, see the discussion at the beginning of section 3.2. In this respect, the differential equation is the most direct method to determine these matrix elements. Finally, the fact that the space of solutions with local exponent β_2 is uni-dimensional is due to the fact that all the matrix elements, even those at the first level, are known. Indeed in this case $\vec{\alpha}_M^{(t)} = (s-b)\vec{\omega}_2$. It is a semi-degenerate anti-fundamental field and the corresponding matrix elements (5.15) involve this field can be evaluated for any p .

5.2.3 The case with $\Phi_{s\vec{\omega}_1-b\vec{\omega}_2}$

We consider now the conformal block:

$$\mathcal{B}_M(z) \stackrel{\text{def}}{=} \Phi_L^*(\infty) \begin{array}{c} \Phi_{s\vec{\omega}_1-b\vec{\omega}_2}(1) \quad \Phi_{2111}(z) \\ | \qquad \qquad | \\ \hline M \\ \hline \Phi_R(0) \end{array} \quad (5.21)$$

where we consider the other semi-degenerate field at level two, $\Phi_{s\vec{\omega}_1-b\vec{\omega}_2}$. In this respect it is convenient to use the invariance of the \mathcal{W}_3 conformal blocks under the exchange $\vec{\omega}_1 \leftrightarrow \vec{\omega}_2$ and to consider the following conformal block:

$$\mathcal{B}_M(z) \stackrel{\text{def}}{=} \Phi_L^*(\infty) \begin{array}{c} \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}(1) \quad \Phi_{1211}(z) \\ | \qquad \qquad | \\ \hline M \\ \hline \Phi_R(0) \end{array} \quad (5.22)$$

Note that we have kept the fields $\Phi_{R,L}$ un-exchanged as they are general fields. The computation of the differential equation satisfied by (5.22) is strictly similar to the one

for (4.2), the only difference being the fact that q_1 gets an opposite sign with respect to the previous case (3.1). The resulting sixth order equation has the same pattern of local exponents as in (5.2). Their precise values are given by:

$$\begin{aligned}\alpha_1 &= \frac{1}{3} (2a_{R_2}b + a_{R_1}b), & \alpha_2 &= \frac{1}{3} (3 - a_{R_2}b + a_{R_1}b + 3b^2), & \alpha_3 &= \frac{1}{3} (6 - a_{R_2}b - 2a_{R_1}b + 6b^2), \\ \beta_1 &= \frac{1}{3} (3 + 2b^2 - bs), & \beta_2 &= \frac{1}{3} (6 + 8b^2 - bs), & \beta_3 &= \frac{1}{3} (-b^2 + 2bs), \\ \gamma_1 &= \frac{1}{3} (-6 + 2a_{L_1}b + a_{L_2}b - 8b^2), & \gamma_2 &= \frac{1}{3} (-3 - a_{L_1}b + a_{L_2}b - 5b^2), & \gamma_3 &= \frac{1}{3} (-a_{L_1}b - 2a_{L_2}b - 2b^2),\end{aligned}\quad (5.23)$$

and correspond to the following fusion rules: in the s, t, u channels:

- s -channel:

$$\text{Channel 1,2 } (\alpha_1): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R - b\vec{\omega}_2 \quad (5.24)$$

$$\text{Channel 3,4 } (\alpha_2): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R + b \left(\vec{\omega}_2 - \vec{\omega}_1 \right) \quad (5.25)$$

$$\text{Channel 5,6 } (\alpha_3): \quad \vec{\alpha}_M^{(s)} = \vec{\alpha}_R + b\vec{\omega}_1 \quad (5.26)$$

- t -channel:

$$\text{Channel 1,2,3 } (\beta_1): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) - b \left(\vec{\omega}_1 - \vec{\omega}_2 \right) \quad (5.27)$$

$$\text{Channel 4 } (\beta_2): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) + b\vec{\omega}_1 \quad (5.28)$$

$$\text{Channel 5,6 } (\beta_3): \quad \vec{\alpha}_M^{(t)} = \left(-b\vec{\omega}_1 + s\vec{\omega}_2 \right) - b\vec{\omega}_2 \quad (5.29)$$

- u -channel:

$$\text{Channel 1,2 } (\gamma_1): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* - b\vec{\omega}_2 \quad (5.30)$$

$$\text{Channel 3,4 } (\gamma_2): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* + b \left(\vec{\omega}_2 - \vec{\omega}_1 \right) \quad (5.31)$$

$$\text{Channel 5,6 } (\gamma_3): \quad \vec{\alpha}_M^{(u)} = \vec{\alpha}_L^* + b\vec{\omega}_1 \quad (5.32)$$

The above values are consistent with the expected fusion rules and the analysis of the degeneracy pattern is strictly analogous to the one done above for the case (4.2).

6 Summary and discussion

The main motivation of our study lies in the fact that, in $\mathcal{W}_{N \geq 3}$ theories, the general matrix element of a primary field between two descendant states is not expressed solely in terms of the primary 3-point function but involves also an infinite set of new independent basic matrix elements. This greatly limits the available information on correlation functions. Another manifestation of this is that the AGT correspondence [8, 9] for \mathcal{W}_N theories allows to construct matrix elements only for the fields with highest-weights proportional either to ω_1 or to ω_{N-1} fundamental weights of \mathfrak{sl}_N . In this case, the correspondence between 2-dimensional conformal field theory and 4-dimensional supersymmetric gauge theories, as

proposed in [8], is available, and \mathcal{W}_N conformal blocks are equal to Nekrasov instanton partition functions [9]. See [19, 24–27] for recent works towards a more general analysis.

In this paper we focused our attention on the field $\Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}$ of the \mathcal{W}_3 Toda conformal field theory. We showed that this field is a second level semi-degenerate field and we found the corresponding null-vector conditions (3.16). These conditions allow for the computation of all except one matrix elements involving this field: the basis elements (5.15) with $p = 2, 3, \dots$ can indeed be computed as a function of the matrix element (5.15) with $p = 1$. The (5.19) is an example of such relations. Moreover we derived the differential equation obeyed by the conformal block containing a fully-degenerate fundamental fields $\Phi_{-b\vec{\omega}_1}$, the semi-degenerate field $\Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}$ and two general fields Φ_R and Φ_L . We computed the local exponents of this Fuchsian equation and we related the corresponding multiplicities to the number of undetermined matrix elements (5.15). Interestingly we also argued that the field $\Phi_{-2b\vec{\omega}_1+s\vec{\omega}_1}$ is a semi-degenerate field at level 3 and the associated matrix elements (5.15) with $p = 3, 4, \dots$ can be computed, via the differential equation, in terms of the ones with $p = 1, 2$.

Our results demand further investigations. To begin with, it would be interesting to find the monodromy group associated to the sixth-order differential systems. This would allow the definition of local correlation functions, to compare with the ones computed in [6] by completely different methods, and the determination of \mathcal{W}_3 structure constants that are, at the present, unknown. Moreover, it would be interesting to understand how to recover our results for the semi-degenerate level-2 fields using the AGT correspondence. In this respect, the case of central charge $c = 2$ can be tackled with the methods proposed in [28].

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A Details of the derivation

Besides the conformal block $\mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\})$, we need to consider the two functions:

$$\begin{aligned} W_{-1}^{(2)} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) &= \langle \Phi_L^*(z_L) [W_{-1} \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}](z_2) \Phi_{-b\vec{\omega}_1}(z_1) \Phi_R(z_R) \rangle, \quad (\text{A.1}) \\ [W_{-1}^2]^{(2)} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) &= \langle \Phi_L^*(z_L) [W_{-1}^2 \Phi_{-b\vec{\omega}_1+s\vec{\omega}_2}](z_2) \Phi_{-b\vec{\omega}_1}(z_1) \Phi_R(z_R) \rangle. \end{aligned}$$

Analogously to (2.22), the global conformal invariance fixes the coordinate dependence:

$$\begin{aligned} W_{-1}^{(2)} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) &= (z_L - z_1)^{-2h_1} (z_L - z_R)^{h_1-h_R+h_2+1-h_L} \times \\ &\times (z_L - z_2)^{h_1+h_R-h_2-1-h_L} (z_2 - z_R)^{-h_1-h_R-h_2-1+h_L} H(z), \end{aligned}$$

$$[W_{-1}^2]^{(2)} \mathcal{B}_M \left(L, 2, 1, R \right) (\{z_X\}) = (z_L - z_1)^{-2h_1} (z_L - z_R)^{h_1 - h_R + h_2 + 2 - h_L} \times \\ \times (z_L - z_2)^{h_1 + h_R - h_2 - 2 - h_L} (z_2 - z_R)^{-h_1 - h_R - h_2 - 2 + h_L} H_1(z),$$

where

$$H(z) \stackrel{\text{def}}{=} W_{-1}^{(2)} \mathcal{B}_M(z), \quad H_1(z) \stackrel{\text{def}}{=} [W_{-1}^2]^{(2)} \mathcal{B}_M(z), \quad (\text{A.2})$$

and z is given in (2.23). We recall that the values of (h_1, q_1) and (h_2, q_2) , characterizing the fields at position z_1 and z_2 , are given respectively in (3.1) and in (3.13). Using the (3.16) for the semi-degenerate field at z_2 , one has the relation

$$c_{\{2;\emptyset\}}^{\text{sing}} L_{-2}^{(2)} \mathcal{B}_M(z) + c_{\{\emptyset;2\}}^{\text{sing}} W_{-2}^{(2)} \mathcal{B}_M(z) + c_{\{1,1;\emptyset\}}^{\text{sing}} [L_{-1}^2]^{(2)} \mathcal{B}_M(z) + c_{\{1;1\}}^{\text{sing}} L_{-1}^{(2)} H(z) + H_1(z) = 0. \quad (\text{A.3})$$

where c_I^{sing} are the components of the vector c^{sing} given in (3.14).

The functions $L_{-2}^{(2)} \mathcal{B}_M(z)$ and $[L_{-1}^2]^{(2)} \mathcal{B}_M(z)$, related to conformal blocks involving pure Virasoro descendants, can be expressed in terms of differential operators acting on $\mathcal{B}_M(z)$:

$$L_{-2}^{(2)} \mathcal{B}_M(z) = \frac{(z-2)z}{z-1} \mathcal{B}'_M(z) + \left[h_1 + 2h_R + h_2 - h_L + \frac{h_1}{(z-1)^2} \right] \mathcal{B}_M(z), \quad (\text{A.4})$$

$$[L_{-1}^2]^{(2)} \mathcal{B}_M(z) = z^2 \mathcal{B}''_M(z) + 2z(h_1 + h_R + h_2 - h_L + 1) \mathcal{B}'_M(z) + \\ + (h_1 + h_R + h_2 - h_L)(h_1 + h_R + h_2 - h_L + 1) \mathcal{B}_M(z), \quad (\text{A.5})$$

while the function $L_{-1}^{(2)} H(z)$ is easily expressed as

$$L_{-1}^{(2)} H(z) = -(h_1 + h_R + h_2 - h_L + 1) H(z) - zH'(z). \quad (\text{A.6})$$

Less direct is the computation of $W_{-2}^{(2)} \mathcal{B}_M(z)$ term. Using five \mathcal{W} -Ward identities for $\mathcal{B}_M(z)$ together with the three null-vector conditions (3.2), (3.3) and (3.4) for the field $\Phi_{-b\bar{\omega}_1}(z_1)$ allows to express $W_{-2}^{(2)} \mathcal{B}_M(z)$ in terms of $\mathcal{B}_M(z)$ and $H(z)$. The resulting expressions is

$$W_{-2}^{(2)} \mathcal{B}_M(z) = -2H(z) - \frac{12q_1 z^2 \mathcal{B}''_M(z)}{h_1(5h_1+1)} - \frac{3q_1 z(h_1(9z-7) + 5z-3) \mathcal{B}'_M(z)}{h_1(5h_1+1)(z-1)} - \\ \times \left[\frac{q_1(5h_1-11h_1z-6h_Lz+7z-1)}{(5h_1+1)(z-1)} + \frac{6q_1(h_1h_R+h_R-h_2z-h_1h_2z-h_Lz)}{h_1(5h_1+1)(z-1)} \right. \\ \left. + \frac{2h_1q_1(11z^2-16z+5)}{(5h_1+1)(z-1)^2} + \frac{6h_2(h_1+1)q_1(z-2)z}{h_1(5h_1+1)(z-1)^2} + q_R + q_2 - q_L \right] \mathcal{B}_M(z). \quad (\text{A.7})$$

Finally we have to express the function $H(z)$ and $H_1(z)$ in terms of the differential operator acting on $\mathcal{B}_M(z)$. This can be done by using the Ward identities (4.4)–(4.8). We obtain the following two identities:

$$\frac{1}{(z-1)^2 z^2} H(z) + g_0 \mathcal{B}_M(z) + g_1 \mathcal{B}'_M(z) + g_2 \mathcal{B}''_M(z) + g_3 \mathcal{B}'''_M(z) = 0, \quad (\text{A.8})$$

and

$$\frac{1}{(z-1)^2 z^2} H_1(z) + \tilde{g}_0 \mathcal{B}_M(z) + \tilde{g}_1 \mathcal{B}'_M(z) + l_0 H(z) + l_1 H'(z) + l_2 H''(z) + l_3 H'''(z) = 0. \quad (\text{A.9})$$

The coefficients in the above equations read respectively:

$$g_0 = \frac{3(3h_1 - 1)h_R q_1(2 - 3z)}{2h_1(5h_1 + 1)(z - 1)^2 z^3} - \frac{(q_R(z - 1)^2 - q_L z(z - 1)^2 - q_2(2z - 1)z)}{(z - 1)^3 z^3} \\ + \frac{3h_2 q_1(h_1(9z - 7) - 3z - 3)}{2h_1(5h_1 + 1)(z - 1)^3 z^2} + \frac{3h_L q_1(h_1(10z - 7) + 2z - 3)}{2h_1(5h_1 + 1)(z - 1)^2 z^2} \\ + \frac{q_1(h_1(31 - 40z) - 8z + 11)}{2(5h_1 + 1)(z - 1)^2 z^2}, \quad (\text{A.10})$$

$$g_1 = -\frac{3q_1(44z^2 - 53z + 12)}{2(5h_1 + 1)(z - 1)^2 z^2} - \frac{12q_1(h_R(z - 1) - h_2 z)}{h_1(5h_1 + 1)(z - 1)^2 z^2} \\ - \frac{3q_1(20z^2 - 25z + 4)}{2h_1(5h_1 + 1)(z - 1)^2 z^2} + \frac{12h_L q_1}{h_1(5h_1 + 1)(z - 1)z}, \quad (\text{A.11})$$

$$g_2 = \frac{12q_1(1 - z)(5z - 3)}{h_1(5h_1 + 1)(z - 1)^2 z}, \quad g_3 = \frac{16q_1(1 - z)}{h_1(h_1 + 1)(5h_1 + 1)(z - 1)}, \quad (\text{A.12})$$

and

$$\tilde{g}_0 = \frac{9q_1 q_2(h_1(9z - 8) - 3z - 8)}{2h_1(5h_1 + 1)(z - 1)^4 z} + \frac{(h_1 + h_R)\kappa(1 - 2z)}{(z - 1)^3 z^2} - \frac{h_L \kappa(1 - 2z)}{(z - 1)^3 z^2} \\ - \frac{h_2 \kappa(z^2 - 3z + 1)}{(z - 1)^4 z^2}, \quad (\text{A.13})$$

$$\tilde{g}_1 = \frac{36q_1 w_3}{h_1(5h_1 + 1)(z - 1)^3} + \frac{\kappa(1 - 2z)}{(z - 1)^3 z}, \quad (\text{A.14})$$

$$l_0 = \frac{q_1(h_1 z(31 - 40z) + 9h_R(2 - 3z))}{2(5h_1 + 1)(z - 1)^2 z^3} - \frac{9(h_2 + 1)q_1(z + 1)}{2h_1(5h_1 + 1)(z - 1)^3 z^2} \\ - \frac{(q_R(z - 1)^2 - q_L z(z - 1)^2 + q_2 z(1 - 2z))}{(z - 1)^3 z^3} + \frac{3h_2 q_1(9z - 7)}{2(5h_1 + 1)(z - 1)^3 z^2} \\ - \frac{q_1(4z^2 - 23z + 16)}{(5h_1 + 1)(z - 1)^3 z^2} + \frac{3q_1(h_R(3z - 2) + h_L(2z - 3)z + h_1 h_L(10z - 7)z)}{2h_1(5h_1 + 1)(z - 1)^2 z^3}, \quad (\text{A.15})$$

$$l_1 = \frac{12q_1(h_L z - h_R)}{h_1(5h_1 + 1)(z - 1)z^2} - \frac{3q_1(44z^2 - 53z + 12)}{2(5h_1 + 1)(z - 1)^2 z^2} \\ - \frac{3q_1(20z^2 - 33z + 4)}{2h_1(5h_1 + 1)(z - 1)^2 z^2} + \frac{12h_2 w_1}{h_1(5h_1 + 1)(z - 1)^2 z}, \quad (\text{A.16})$$

$$l_2 = \frac{12q_1(3 - 5z)}{h_1(5h_1 + 1)(z - 1)z}, \quad l_3 = -\frac{16q_1}{h_1(h_1 + 1)(5h_1 + 1)}. \quad (\text{A.17})$$

Using (A.8) and (A.9), we are able to express $H(z)$ and $H_1(z)$ in terms of $\mathcal{B}_M(z)$ and its derivatives (up to 6th order). Using these results in (A.3), we finally get the 6th order differential equation for $\mathcal{B}_M(z)$.

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